The Basics of Bayesian Inference

POST 8000 – Foundations of Social Science Research for Public Policy

Steven V. Miller

Department of Political Science

Goal for Today

Introduce students to the basics of Bayesian inference.

"Frequentist" Inference and Research Design

You should be familiar with our discussion of research design and quantitative analysis to this point.

- Concepts, measures, variables, et cetera.
- Research design and the logic of control.
- Random sampling of the population (i.e. inferential statistics).
- Regression (linear or logistic) as (ideally) estimating cause and effect with observational data.

“Frequentist” Inference and Research Design

We summarize inference as follows.

- If our regression coefficient is at least $\pm$ 1.96 standard errors from zero, we reject the null hypothesis.
- The regression coefficient is “statistically significant” in support of a hypothesis.
- The regression coefficient emerges as best estimate of causal effect.

We know this because central limit theorem tell us this is true.

“Statistically Significant” Frequentist Inference

The simplicity of “statistically significant” is powerful and deceptive.

- When $z = 1.96$, we would observe a coefficient that far from zero five times in 100 random samples, on average.

Notice more carefully what’s happening.

- We assume a fixed parameter (here: the null).
- We make statements of relative frequencies of extreme results under it.

“Statistically Significant” Frequentist Inference

Does that really make sense?

- Central limit theorem says it’s true.

However, it depends on two things we routinely don’t have.

1. Known population parameters
2. Repeated sampling

Probability and Frequentist Inference

Objectivist probability is the foundation for classical statistics.

Objectivist Probability

For example, the probability of a tossed coin landing heads up is a characteristic of the coin itself.

- By tossing it infinitely and recording the results, we can estimate the probability of a head.

Formally:

$$Pr(A) = \lim_{n \to \infty} \frac{m}{n}$$

...where:

- *n*: number of trials
- *m*: number of times we observe event *A*
- *A*: outcome in question (here: a coin landing heads up).

Objectivist Probability and Frequentist Inference

We can understand why classical statistics is **frequentist** and **objectivist**.

- Frequentist: probability is a long-run relative *frequency* of an event.
- Objectivist: probability is a characteristic of the object itself.
    - e.g. cards, dice, coins, roulette wheels.

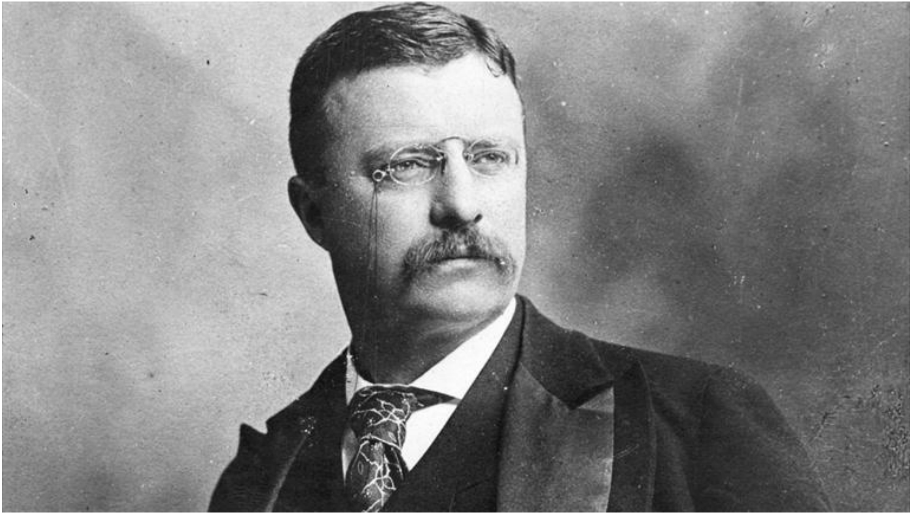
Bayesian Probability

Bayesian probability statements are states of mind about the states of the world and not states of the world, per se.

- It is a *belief* of some event occurring.
- It is characterized as *subjective* probability accordingly.

There are constraints, but nonetheless a substantial amount of variation allowed on probabilistic statements.

Bayesian Probability: An Unintuitive Application



What is the probability that Teddy Roosevelt is the 25th U.S. President?

Bayesian Probability

A Bayesian approach:

- What is my degree of belief that statement is true?

A frequentist approach:

- Well, was he or wasn't he?

Since there is only one experiment for this phenomenon, the frequentist probability is either 0 or 1.

- The phenomena is neither standardized nor repeatable.

Bayesian Probability

Even greater difficulties arise for future events. For example:

- What is the probability of a terrorist attack in the U.S. that claims at least 50 lives in the next five years?
- What is the probability of a war between the U.S. and Iran?
- What is the probability the Democrats win the House in 2020? The Senate? The White House? All three?

Bayesian Inference

These are all perfectly legitimate and interesting questions.

- However, frequentist inference offers no helpful answer.

Bayesian inference does offer a helpful route in **Bayes' theorem**.

Bayesian Inference

The probability of event *A* given *B* for a continuous space:

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

With only two possible outcomes: *A* and ~A

$$p(A|B) = \frac{p(B|A)p(A)}{p(B|A)p(A) + p(B| \sim A)p(\sim A)}$$

Bayesian Inference: An Illustration with Pregnancy Tests

Suppose a woman wants to know if she's pregnant.

- She acquires a name-brand test that purports to be 90% reliable.
  - i.e. if you're pregnant, you'll test positive 90% of the time.
- It gives false positives 50% of the time.
  - i.e. if you're not pregnant, you'll test positive 50% of the time.
- Suppose the probability of getting pregnant after a sexual encounter is $p = .15$
  - *Note*: this is just one number I found. I'm not that kind of doctor.

Bayesian Inference: An Illustration with Pregnancy Tests

Suppose the woman tested positive.

- She knows her test purports 90% accuracy in testing positive, given she is pregnant.
- *She wants to know if she's pregnant, given she tested positive*.

Bayesian Inference: An Illustration with Pregnancy Tests

We are interested in p(preg | test +). We know the following:

- p(test + | preg) = .90
- p(preg) = .15 (conversely: p(~preg) = .85).
- p(test + | ~preg) = .50.

We have this derivation of Bayes' theorem.

$$p(\text{preg}|\text{test +}) = \frac{p(\text{test +}|\text{preg})p(\text{preg})}{p(\text{test +}|\text{preg})p(\text{preg}) + p(\text{test +}|\sim\text{preg})p(\sim\text{preg})}$$

Bayesian Inference: An Illustration with Pregnancy Tests

We can now answer $p(\text{preg} \mid \text{test} +)$.

$$p(\text{preg}|\text{test} +) = \frac{(.90)(.15)}{(.90)(.15) + (.50)(.85)} = \frac{.135}{.135 + .425} = .241$$

This is far from the belief you'd get from "90% accuracy" and a single positive test.

Posterior Probability

However, this quantity is important for Bayesians in its own right: a **posterior probability**.

- It's an updated probability of event A (being pregnant) after observing the data B (the positive test).
- She has a prior belief of being pregnant (p = .15), which is now updated to p = .241.

Does this mean the woman is really not pregnant?

Posterior Probability

She should take the updated posterior probability as "prior information" (i.e. p(preg) = .241, and p(~preg) = .759) and take another test.

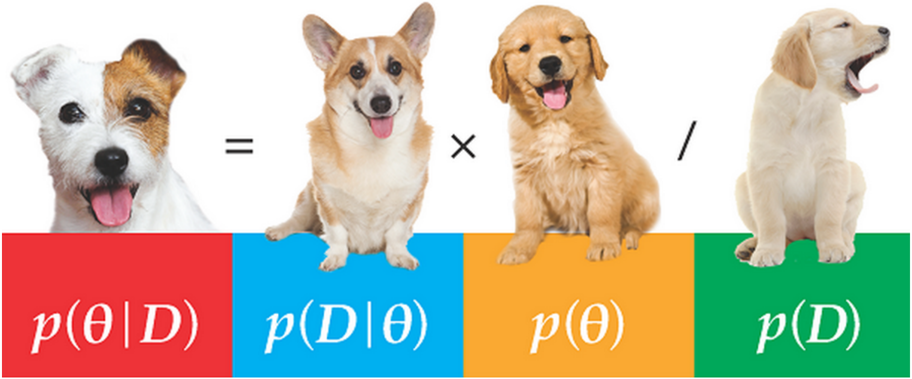
- Assume, again, she tested positive.

$$p(\text{preg}|\text{test +}) = \frac{(.90)(.241)}{(.90)(.241) + (.50)(.759)} = \frac{.216}{.216 + .379} = .363$$

Posterior Probability

In other words, keep repeating tests until you're convinced, but don't begin agnostic each time.

Bayesian Inference



Bayesian inference uses this uncontroversial imputation of conditional probability as a foundation for statistical inference.

Bayesian Inference

$$p(\theta|D) = \frac{p(D|\theta)p(\theta)}{p(D)}$$

The probability of an unknown parameter, given the data we observed is equal to the joint probability of θ and D, divided by the probability of D.

Bayesian Inference, Simplified

We say the posterior distribution (i.e. likelihood of the unknown parameter given the data) is *proportional to* the likelihood of the data multiplied by our prior expectations of it.

$$\text{Posterior} \propto \text{Likelihood} \times \text{Prior}$$

...where $\propto$ means “is proportional to” in symbol form.

The Benefits of Bayesian Inference

Inference is much less clunky.

- Frequentist: what is probability of data, given some (fixed, unobservable, implausible, always “null”) parameter?
- Bayesian: what parameters are plausible, given the data?

Explicitly models/incorporates prior beliefs.

- No effect is truly “null.”
- Allows for some novel competitive hypothesis testing (see: Western and Jackman, 1994).
- Acknowledges prior distributions (whereas frequentist likelihood models sweep them under the rug).

Allows for greater flexibility in model summary (posterior distributions).

- No ad hoc standard error corrections/approximations.
- Posterior distribution comes free with the analysis.

Greater appreciation in getting the best estimate of a parameter, with uncertainty.

The Drawbacks of Bayesian Inference

Bayesian inference is computationally demanding.

- Retort: Supercomputing helps, but this is still true.
- Silver lining: You get more out of the model, and greater insight to potential problems in the model.

Bayesian inference is "subjective" while frequentist inference is "objective."

- Retort: making prior beliefs explicit allows greater clarity/transparency.
- Prior distributions are also implicit in frequentist likelihood models. We just sweep them under the rug.

Prior beliefs are "deck-stacking" in support of a hypothesis.

- Retort: this is why we have sensitivity analyses.
- Again: prior distributions are made explicit.

Conclusion

Bayesians highlight how many liberties we can take with our research design if we're not careful.

- Inference is kind of "backward." You're not getting the exact answer to the question you're asking.
- Prior beliefs in frequentist models are implicit and never explicit.
- Inference can be summarized as posterior distributions, given a model of the data.

Table of Contents